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LIPSCHITZ CONTINUITY OF THE TIME CONSTANT FOR CONTINUUM PERCOLATION

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Abstract. We consider the Boolean model of continuum percolation, where points are placed in $\mathbb{R}^d$ by a Poisson point process and pairs of points with distance at most 1 are connected by an edge. The time constant is the limiting ratio of the chemical distance (i.e. graph distance) to the Euclidean distance for pairs of distant connected points. Yao, Chen, and Guo [27] established the existence of a time constant in the supercritical regime. We show that above the critical intensity, the time constant is a Lipschitz continuous function of the intensity. The proof adapts a recent argument of Can, Nakajima, and Nguyen [3] to the continuous setting.

1. Introduction

Given a point set $\mathcal{X} \subset \mathbb{R}^d$, we denote by $G(\mathcal{X})$ the graph with vertex set $\mathcal{X}$ and edge set consisting of $\{x, y\} \subset \mathcal{X}$ such that $\|x - y\| \leq 1$. Let X_λ denote the homogeneous Poisson point process of intensity λ on $\mathbb{R}^d$. We consider the infinite random geometric graph $G(X_\lambda)$, which we abbreviate by G_λ. The critical intensity $\lambda_c = \lambda_c(d)$ is defined by

$$\lambda_c = \inf \{\lambda > 0 : \mathbb{P}(G_\lambda \text{ contains an infinite component}) > 0\}.$$

When $d \geq 2$, $0 < \lambda_c < \infty$, and if $\lambda > \lambda_c$ (the supercritical regime) then G_λ almost surely has a unique infinite component, which we denote by C_∞.

We define a natural metric on $X_\lambda \subset G_\lambda$ by taking the distance between two vertices to be the least number of edges in a path between them. For all $x, y \in G_\lambda$, the *chemical distance* between x and y is

$$d_{G_\lambda}(x, y) := \inf\{|\pi| : \pi \text{ is a path in } G_\lambda \text{ from } x \text{ to } y\}$$

where $|\pi|$ is the number of edges in π. Yao, Chen, and Guo [27] prove the existence of a *time constant* for the chemical distance in supercritical continuum percolation: for each $\lambda > \lambda_c(d)$, there exists a constant $\mu_\lambda(d) \geq 1$ such that

$$\lim_{x\to\infty} \frac{D_\lambda(0, x)}{\|x\|} = \lim_{x\to\infty} \frac{\mathbb{E}[D_\lambda(0, x)]}{\|x\|} = \mu_\lambda(d) \qquad \text{a.s.}$$

where $D_\lambda(x, y) := d_{G_\lambda}(\mathring{x}, \mathring{y})$ and $\mathring{x}$ is the random point of the infinite cluster C_∞ of G_λ that is closest to x in Euclidean distance. Informally, this means that the chemical distance between far away points on an infinite supercritical cluster is very likely approximately μ_λ times the Euclidean distance.

Our main result concerns the continuity of the time constant for continuum percolation. We prove the time constants are Lipschitz continuous in λ in the following sense.

Theorem 1. *Fix $d \geq 2$. For all $\lambda_0 > \lambda_c(d)$, there exists $C(\lambda_0, d)$ such that*

$$|\mu_\lambda(d) - \mu_{\lambda'}(d)| \leq C(\lambda_0, d)|\lambda - \lambda'|$$

for each $\lambda, \lambda' \in [\lambda_0, \infty)$.

Interestingly, the proof actually yields that there is some $C(\lambda_0, d)$ such that $|\mu_\lambda(d) - \mu_{\lambda'}(d)| \leq C(\lambda_0, d)|\log(\lambda/\lambda')|$. That is, the function $t \mapsto \mu_{e^t}$ is also Lipschitz on $[t_0, \infty)$ for $t_0 > \log \lambda_c$.

1.1. **Proof sketch.** In order to prove Lipschitz continuity of μ_λ, it suffices to prove a bound of the form $|\mathbb{E}D_\lambda(0, x) - \mathbb{E}D_{\lambda'}(0, x)| \leq C\|x\||\lambda - \lambda'|$, where $\|x\|$ is the Euclidean distance. Here and below the constant C

is allowed to depend on λ_0 but not λ, λ', or x. First, we define another quantity $\widetilde{D}_\lambda(0,x)$ which approximates $D_\lambda(0,x)$ well, so that it will suffice to prove a bound of the form $|\mathbb{E}\widetilde{D}_\lambda(0,x) - \mathbb{E}\widetilde{D}_{\lambda'}(0,x)| \leq C\|x\||\lambda - \lambda'|$. This random variable $\widetilde{D}_\lambda(0,x)$ will have the advantage that it is a bounded monotone function of the Poisson process X_λ restricted to a finite volume region of $\mathbb{R}^d$. This implies that $\mathbb{E}\widetilde{D}_\lambda(0,x)$ is differentiable in λ (in fact we prove a Russo-type formula for the derivative) and so it will suffice for us to prove a bound of the form $\frac{d}{d\lambda}\mathbb{E}\widetilde{D}_\lambda(0,x) \leq C\|x\|$. The Russo-type formula will allow us to write the left-hand side as a sum over points of the Poisson process, where each summand corresponds to the effect on $\widetilde{D}_\lambda(0,x)$ of removing that point. We will see that the nonzero summands are the points which lie on the geodesic $\widetilde{\pi}$ — a sequence of points of X_λ which "realize" $\widetilde{D}_\lambda(0,x)$. Then, by constructing an appropriate high-probability event, we will see that this sum can be upper bounded (up to $o(\|x\|)$ error) by

$$\frac{1}{\lambda}\mathbb{E}\left[\sum_{q\in\widetilde{\pi}} d_{G_\lambda\setminus\{q\}}(p(q),s(q))\mathbb{1}\left\{d_{G_\lambda\setminus\{q\}}(p(q),s(q)) < \infty\right\}\right],$$

where $p(q)$ and $s(q)$ are respectively the vertices immediately preceding and succeeding q in the geodesic $\widetilde{\pi}$. These summands have good tails by Lemma 3.4 of Yao, Chen, and Guo [27], and they enjoy a particular spatial independence property. Moreover, the length of $\widetilde{\pi}$ is with very high probability at most linear in $\|x\|$. Therefore, after an appropriate discretization, we can apply the lattice animals method of Can, Nakajima, and Nguyen [3] to conclude that this sum is at most linear in $\|x\|$, as desired.

2. Previous Work

The first results on time constants appear in the context of First Passage Percolation (FPP) on graphs, where edges of a graph are given i.i.d. random weights, and the random metric considered is the minimum total weight of a path between two points; this metric is called the *passage time*. The chemical distance of percolation can be viewed as a special case of the passage time where all weights lie in $\{1,\infty\}$ (although many results in FPP assume that the weights are almost surely finite, or an even stronger integrability condition).

The first regularity result for the time constant for FPP on $\mathbb{Z}^2$ appeared in Cox [6] under a uniform integrability assumption on the weight distribution. Still restricted to dimension two, Cox and Kesten [8] removed the integrability condition by considering geodesics for truncated passage times. They introduced the idea of bypassing edges with weights that are too large, and established control over the length of these bypasses. Kesten [17] extended this result to $\mathbb{Z}^d, d \geq 2$ by allowing the truncation to be large enough to control the size of the clusters of closed edges. However, this technique still does not allow for edges with infinite weight. A new approach by Garet, Marchand, Procaccia, and Théret [13] established continuity of the map from the underlying distribution to the time constant for general laws on $[0,+\infty]$ without any moment assumption. Dembin [10] improved the regularity result to a log-Lipschitz result for a generalized percolation model in which first-passage percolation occurs in a random environment. Building upon this work, Cerf and Dembin [4] refined the approach by considering a multi-scale renormalization and proved Lipschitz continuity of the time constant in generalized first-passage percolation. The main ideas were to prove the average size of a bypass is small and to properly choose the different scales of the renormalization process.

Keeping the same approach of looking at bypasses in "good" boxes under the same model and coupling, Can, Nakajima, and Nguyen [3] also established Lipschitz continuity of the time constant in generalized FPP. The insight here is to use lattice animals and a one-step renormalization with Russo's formula to bound the length of the detour caused by removing an edge in the geodesic.

Our proof is inspired by [3] but differs in a few ways. First, the definition of our approximation $\widetilde{D}_\lambda(0,x)$ differs from that of the truncated passage time $T_M^{\Lambda_K}$ in [3]. In particular, $\widetilde{D}_\lambda(0,x)$ only "uses closed edges"

at the beginning and end of the path, which simplifies some technical arguments.[1] Next, arguments about the "effective radius" are replaced by assuming uniqueness events across the whole relevant region in order to ensure connectivity of the endpoints of the geodesic even when an interior point is removed; we then directly bound the length of a bypass around a deleted site assuming this robust connectivity.[2] Finally, as the space we consider is continuous, we must introduce a discretization to apply the lattice animal method in [3]; some care must be taken here to maintain the desired independence property when bundling points together.

3. Preliminaries

We recall some results for point processes and continuum percolation. Recall that throughout, X_λ is the Poisson point process of intensity λ on $\mathbb{R}^d$, and $G_\lambda = G(X_\lambda)$ is the associated geometric random graph.

The following result will be used repeatedly in order to bound expectations of sums indexed by points of our Poisson process.

Theorem 2 (Theorem 4.1, [20]). [Palm theory for Poisson processes.] *For all non-negative bounded and measurable functions f we have*

$$\mathbb{E}\left[\sum_{x_1,x_2,...,x_k\in X_\lambda} f(x_1,x_2,...,x_k,X_\lambda)\right] = \lambda^k \int_{\mathbb{R}^d} \mathbb{E}\left[f(x_1,...,x_k,X_\lambda\cup\{x_1,...,x_k\})\right]\,dx$$
$$= \lambda^k \int_{\mathbb{R}^d} \mathbb{E}^\lambda_{x_1,...,x_k}\left[f(x_1,...,x_k,X_\lambda)\right]\,dx.$$

In the second line, $\mathbb{E}^\lambda_{x_1,...,x_k}$ denotes expectation with respect to the modified probability measure[3] $\mathbb{P}^\lambda_{x_1,...,x_k}$ defined by

$$\mathbb{P}^\lambda_{x_1,...,x_k}(X_\lambda\in A) := \mathbb{P}(X_\lambda\cup\{x_1,...,x_k\}\in A). \tag{1}$$

We will use a few facts about supercritical continuum percolation that Yao, Chen, and Guo established in [27]. The first is that it is unlikely that the origin is far from the giant component; the second is that it is unlikely to have a large component that is not part of the infinite component.

Lemma 3 (Lemma 3.3, Yao, Chen, and Guo [27]). *Suppose $\lambda>\lambda_c$. Then there exists a constant $c=c(\lambda,d)>0$ such that for all large r,*

$$\mathbb{P}(B(0,r)\cap C_\infty=\emptyset)\le\exp(-cr^{d-1}).$$

Lemma 4 (Lemma 3.1, Yao, Chen, and Guo [27]). *Suppose that $\lambda>\lambda_c$. Then there exists a constant $c=c(\lambda,d)>0$ such that, for any Borel set A in $\mathbb{R}^d$ and for all large k, we have*

$$\mathbb{P}(\text{there exists a component } C\subset G_\lambda\cap[-s/2,s/2]^d, k\le|C|<\infty, C\cap A\neq\emptyset)\le\lambda|A|\exp(-ck^{(d-1)/d}).$$

We also use the following local uniqueness event proven in [25].

Proposition 5 (Proposition 10.13, [25]). *Let $\lambda_0>\lambda_c$. Suppose $(\phi_s, s\ge0)$ is increasing with $(\phi_s/\log s)\to\infty$ as $s\to\infty$, and with $\phi_s\le s$ for all s. Let E_s denote the event that at most one component of $G_\lambda\cap[-s/2,s/2]^d$ has metric diameter at least ϕ_s. Then*

$$\limsup_{s\to\infty}\sup_{\lambda\ge\lambda_0}\phi_s^{-1}\log\mathbb{P}^\lambda[E_s^c]<0.$$

[1] To be sure, many of these simplifications are possible because our setting is simpler in the sense that every edge has length 1, whereas in [3], each edge has a random length.

[2] These adaptations are similar to the method pursued in [14] by Procaccia and the second author in the setting of graphs of polynomial growth. [14] was written concurrently with this article.

[3] There is a sense in which $\mathbb{P}^\lambda_{x_1,...,x_k}$ is the law of X_λ conditional on $x_1,...,x_k\in X_\lambda$. The spatial independence of Poisson processes implies that this conditional distribution is simply equal to the law of $X_\lambda\cup\{x_1,...,x_k\}$. We do not strictly need the interpretation of $\mathbb{P}^\lambda_{x_1,...,x_k}$ as a conditional probability, but the notation will be convenient in some cases.

Remark 1. Note that the above bound is uniform over $\lambda \geq \lambda_0$ once we fix a $\lambda_0 > \lambda_c$. This is a stronger statement than that found in [25], but follows from the proofs given there. Let us briefly sketch which details to check in order to get this stronger statement.

The key point is that in Lemma 10.10 in [25], if we set $\mu = \lambda$, we can insert $\inf_{\lambda \geq \lambda_0}$ into the quantities being bounded—in between $\inf_A$ or $\inf_{A,B}$ and the ratio of probabilities—and the statement of the lemma will still hold (as long as $\lambda_0 > \lambda_c$). To see this, in the proof, set $\lambda' := (\lambda_c + \lambda_0)/2$, and then the final bound written in the proof is at least (const.)$\frac{\lambda - \lambda'}{\lambda} \geq 1 - \frac{\lambda'}{\lambda_0} > 0$ for all $\lambda \geq \lambda_0$. This establishes the stronger version of Lemma 10.10.

Then, in the proofs of Lemmas 10.11 and 10.12, the γ extracted from Lemma 10.10 works for all $\lambda \geq \lambda_0$ (and K' only depends on λ_0 as well). Thus the bounds given there are uniform in $\lambda \geq \lambda_0$. Thus, for the case $d \geq 3$, the bounds given in Proposition 10.13 are uniform in $\lambda \geq \lambda_0$, since these are derived from Lemmas 10.11 and 10.12. In the $d = 2$ case of the proof of Proposition 10.13, $\mathbb{P}^\lambda(E_s)$ is bounded below by crossing events, which are increasing, so again we get bounds which are uniform in $\lambda \geq \lambda_0$.

We also use the following fact that the chemical distance in continuum percolation has good tails. Here and throughout, we write $x \xleftrightarrow{G} y$ to denote that two points x and y are connected in the graph G. Recalling the definition of $\mathbb{P}^\lambda_{0,x}$ from (1), we have:

Lemma 6 (Lemma 3.4, Yao, Chen, and Guo [27])**.** *Suppose that $\lambda_0 > \lambda_c$. Then there exist constants $\rho_1 = \rho_1(d, \lambda_0) > 0$ and $c = c(d, \lambda_0) > 0$ such that, for all $x \in \mathbb{R}^d$, all $t \geq \rho_1 \|x\|$, and all $\lambda \geq \lambda_0$ we have*

$$\mathbb{P}^\lambda_{0,x}(0 \xleftrightarrow{G_\lambda} x, d_{G_\lambda}(0,x) \geq t) \leq \exp(-ct). \tag{2}$$

Remark 2. Again, the above statement is stronger than what is stated in [27] in that the bound holds simultaneously for all $\lambda \geq \lambda_0$. However, this stronger statement follows immediately from the proof in [27] if we use the stronger version of Proposition 10.13 in [25] which we state here as Theorem 5.

4. Approximating the chemical distance

Recall that for $v \in \mathbb{R}^d$, we denote by $\mathring{v}$ the (random) point of the infinite cluster of G_λ which is closest in Euclidean distance to v, and recall that we define $D_\lambda(x,y) := d_{G_\lambda}(\mathring{x}, \mathring{y})$. Our first step is to approximate D_λ by a related random variable $\widetilde{D}_\lambda$, which is a bounded monotone function of the Poisson process X_λ restricted to a finite volume region of $\mathbb{R}^d$. Set $M = M(x) = (\log \|x\|)^2$, and define

$$\widetilde{D}_\lambda(0,x) := \inf\{d_{G_\lambda}(p,q) + M\left(\|p\| + \|x - q\|\right) : p, q \in G_\lambda\} \cup \{M\|x\|\}.$$

It is straightforward to check that, considered as a function of the point process X_λ, $\widetilde{D}_\lambda(0,x)$ is nonincreasing; that is, if $X \subset X'$, then $\widetilde{D}(0,x)[X] \geq \widetilde{D}(0,x)[X']$.

We now introduce notation to keep track of the vertices and edges which realize the infimum in the definition of $\widetilde{D}_\lambda(0,x)$. If $\widetilde{D}_\lambda(0,x) < M\|x\|$, define $\widetilde{0}, \widetilde{x} \in G_\lambda$ to be the points which satisfy

$$\widetilde{D}_\lambda(0,x) = d_{G_\lambda}(\widetilde{0}, \widetilde{x}) + M(\|\widetilde{0}\| + \|x - \widetilde{x}\|), \tag{3}$$

and define $\widetilde{\pi}$ to be the edge path in G_λ from $\widetilde{0}$ to $\widetilde{x}$ with $d_{G_\lambda}(\widetilde{0}, \widetilde{x})$ edges (if there is more than one candidate, choose the one of smallest total Euclidean length). If $\widetilde{D}_\lambda(0,x) = M\|x\|$, set $\widetilde{0} = \widetilde{x} = 0$ and $\widetilde{\pi}$ to be the empty path from 0 to 0.

Note that if any vertex v in $\widetilde{\pi}$ had Euclidean distance greater than $M\|x\|$ from 0 or x, then we would have

$$\widetilde{D}_\lambda(0,x) = |\widetilde{\pi}| + M(\|\widetilde{0}\| + \|x - \widetilde{x}\|) \geq \|v\| + \|x - v\| > M\|x\|,$$

which contradicts the definition of $\widetilde{D}_\lambda(0,x)$. Hence we see that $\widetilde{D}_\lambda(0,x)$ depends only on the Poisson process X_λ restricted to the finite volume region $R(\widetilde{D}) := \{y \in \mathbb{R}^d : \|y\|, \|x - y\| \leq M\|x\|\}$.

Our main goal in this section is to show that $\widetilde{D}_\lambda(0,x)$ is a good approximation to $D_\lambda(0,x)$ (Theorem 8 below). In order to do this, we first need a bound on the tails of $D_\lambda(0,x)$. We prove a slightly stronger statement than necessary so that we can reuse it later in the proof of Theorem 22.

Proposition 7. *There exist constants C_1, C_2, c_3 depending only on λ_0 and d such that for all $t \geq C_1\|x\|$ and all $\lambda \geq \lambda_0$ we have*

$$\mathbb{P}^\lambda(D_\lambda(0,x) + M(\|\mathring{0}\| + \|x - \mathring{x}\|) \geq t) \leq C_2 \exp(-c_3 t^{.99}).$$

Proof. Let $\rho_1(\lambda_0) < \infty$ be as in Theorem 6. Then whenever $t \geq \|x\|$ we have

$$\begin{aligned}
&\mathbb{P}^\lambda\Big(D_\lambda(0,x) + M\|\mathring{0}\| + M\|x - \mathring{x}\| \geq (3\rho_1 + 2)t\Big) \\
&\leq \mathbb{P}^\lambda\left(d_{G_\lambda}(\mathring{0}, \mathring{x}) \geq 3\rho_1 t, \|\mathring{0}\|, \|x - \mathring{x}\| \leq \frac{t}{M}\right) + \mathbb{P}^\lambda\left(\|\mathring{0}\| > \frac{t}{M}\right) + \mathbb{P}^\lambda\left(\|x - \mathring{x}\| > \frac{t}{M}\right) \\
&\leq \mathbb{E}^\lambda\left[\sum_{\substack{p \in B\left(0, \frac{t}{M}\right) \cap X_\lambda \\ q \in B\left(x, \frac{t}{M}\right) \cap X_\lambda}} \mathbb{1}\left\{\mathring{0} = p, \mathring{x} = q, d_{G_\lambda}(p,q) \geq 3\rho_1 t\right\}\right] + 2\mathbb{P}^{\lambda_0}\left(C_\infty \cap B\left(0, \frac{t}{M}\right) = \emptyset\right) \\
&\leq \lambda^2 \int_{\substack{p \in B(0,t/M) \\ q \in B(x,t/M)}} \mathbb{P}^\lambda_{p,q}\big(\mathring{0} = p, \mathring{x} = q, d_{G_\lambda}(p,q) \geq 3\rho_1 t\big) dpdq + 2\exp(-c_2(t/M)^{d-1}).
\end{aligned}$$

(Recall the definition of $\mathbb{P}^\lambda_{p,q}$ from (1).) In the last line we used Theorem 2 and Theorem 3. Since the second summand is $O(-c_2[t/(\log t)^2]^{d-1})$, we now focus on the first term.

Using Cauchy-Schwarz, Theorem 6, and Theorem 3 then gives

$$\begin{aligned}
&\lambda^2 \int_{\substack{p \in B(0,t/M) \\ q \in B(x,t/M)}} \mathbb{P}^\lambda_{p,q}\big(\mathring{0} = p, \mathring{x} = q, d_{G_\lambda}(p,q) \geq 3\rho_1 t\big) dpdq \\
&\leq \lambda^2 \int_{\substack{p \in B(0,t/M) \\ q \in B(x,t/M)}} \sqrt{\mathbb{P}^\lambda_{p,q}\big(\mathring{0} = p, \mathring{x} = q\big)\mathbb{P}^\lambda_{p,q}\big(d_{G_\lambda}(p,q) \geq 3\rho_1 t\big)} dpdq \\
&\leq \lambda^2 \exp\left(-\frac{3\rho_1 c}{2} t\right) \int_{\substack{p \in B(0,t/M) \\ q \in B(x,t/M)}} \mathbb{P}^\lambda(C_\infty \cap B(0, \|p\|) = \emptyset)^{1/4} \mathbb{P}^\lambda(C_\infty \cap B(x, \|x - q\|) = \emptyset)^{1/4} dpdq \\
&\leq \lambda^2 \exp\left(-\frac{3\rho_1 c}{2} t\right) \left(\int_{p \in B(0,t/M)} \mathbb{P}^{\lambda_0}\left(\left(\frac{\lambda_0}{\lambda}\right)^{1/d} C_\infty \cap B(0, \|p\|) = \emptyset\right)^{1/4} dp\right)^2 \\
&\leq \lambda^2 \exp\left(-\frac{3\rho_1 c}{2} t\right) \left(\int_{p \in \mathbb{R}^d} \mathbb{P}^{\lambda_0}\left(C_\infty \cap B\left(0, \left(\frac{\lambda}{\lambda_0}\right)^{1/d} \|p\|\right) = \emptyset\right)^{1/4} dp\right)^2 \\
&\leq \lambda^2 \exp\left(-\frac{3\rho_1 c}{2} t\right) \left(\int_{p \in \mathbb{R}^d} \exp\left(-\frac{c_2}{4}\left[\frac{\lambda^{1/d}\|p\|}{\lambda_0^{1/d}}\right]^{d-1}\right) dp\right)^2 \\
&= \exp\left(-\frac{3\rho_1 c}{2} t\right) \left(\int_{R \in \mathbb{R}^d} \exp\left(-\frac{c_2}{4\lambda_0^{\frac{d-1}{d}}}\|R\|^{d-1}\right) dR\right)^2.
\end{aligned}$$

Note that Theorem 6 applies to all p, q under consideration since for such p, q we have $\|p - q\| \leq 3t$. In both the second and third lines Cauchy-Schwarz is used. In the fourth line we use that X_λ is equal in distribution to $\left(\frac{\lambda_0}{\lambda}\right)^{1/d} X_{\lambda_0}$, and so in particular there is a coupling of X_λ and X_{λ_0} such that $C_\infty(X_\lambda)$ almost surely contains $\left(\frac{\lambda_0}{\lambda}\right)^{1/d} C_\infty(X_{\lambda_0})$. The cancellation of λ^2 in the last line comes from the change of variables $R = \lambda^{1/d} p$.

Thus, if we take any $\infty > C_1 > 3\rho_1$ and any $0 < c_3 < \frac{3\rho_1 c}{2(3\rho_1 + 2)}$ we can find a C_2 sufficiently large that

$$\mathbb{P}^\lambda(D_\lambda(0,x) + M(\|\mathring{0}\| + \|x - \mathring{x}\|) \geq t) \leq C_2 \exp(-c_3 t^{.99}),$$

as desired. □

We now come to the main result of this section.

Proposition 8. *For every $\lambda > \lambda_c$,*

$$\mathbb{E}|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)| = o(\|x\|),$$

(where the rate of convergence implicit in the little-o notation may depend on λ).

We begin by constructing an event A that happens with high probability, and then we show an almost sure bound on $|\widetilde{D}_\lambda(0,x) - D_\lambda(0,x)|$ on the event A via persistent use of the triangle inequality. Recall that we defined $R(\widetilde{D}) := \{y \in \mathbb{R}^d : \|y\|, \|x-y\| \le M\|x\|\}$.

Proof of Theorem 8. For $\lambda > \lambda_c$, let ρ_1 be as in Theorem 6.

We define the following events:

$$A_1 := \left\{ \forall p, q \in G_\lambda \cap R(\widetilde{D}),\ \text{either } p \overset{G_\lambda}{\longleftrightarrow\!\!\!\!\!\!/\ \ } q \text{ or } d_{G_\lambda}(p,q) \le \rho_1 \max(\|p-q\|, M) \right\}$$

$$A_2 := \{\forall v \in G_\lambda \cap R(\widetilde{D}),\ \text{either } v \in C_\infty \text{ or } |C(v)| \le \|x\|/2\}$$

$$A_3 := \{\|\mathring{0}\|, \|x - \mathring{x}\| \le M\},$$

and let $A = A_1 \cap A_2 \cap A_3$. On the event A, all pairs which are connected in G_λ have chemical distance not too large; there are no large components other than the infinite one; and both 0 and x are not too far from the infinite component.

Claim 9. $\mathbb{P}(A^c) = o(M^{-2})$ *(where the constants implicit in the little o notation may depend on λ).*

Proof of Theorem 9: By Theorem 6 (and Theorem 2), the probability that some pair of points in $R(\widetilde{D})$ have large chemical distance is $O(M^{2d}\|x\|^{2d})\exp(-cM)$; by Theorem 4, the probability that there is a large component that is not the infinite component is $O((M\|x\|)^d \exp(-c\|x\|^{\frac{d-1}{d}})$; and by Theorem 3, the probability that either 0 or x is more than distance M from the infinite component is $O\left(\exp(-cM^{d-1})\right)$. ■

Next, we want to show an almost-sure bound

$$\mathbb{1}_A|\widetilde{D}_\lambda(0,x) - D_\lambda(0,x)| = o(\|x\|). \tag{4}$$

Assume that the event A holds.

Let us first take for granted that $\widetilde{0}, \mathring{0}, \widetilde{x}, \mathring{x}$ are all connected in G_λ (we establish this in Theorem 10 below). Then the chemical distance condition guaranteed by A_1 shows us that:

$$\begin{aligned} d_{G_\lambda}(\mathring{0},\mathring{x}) - d_{G_\lambda}(\widetilde{0},\widetilde{x}) &\le d_{G_\lambda}(\mathring{0},\widetilde{0}) + d_{G_\lambda}(\mathring{x},\widetilde{x}) \\ &\le \rho_1 \left[\max\left(\|\mathring{0}-\widetilde{0}\|, M\right) + \max\left(\|\mathring{x}-\widetilde{x}\|, M\right)\right] \\ &\le \rho_1 \left[\|\mathring{0}\| + \|\widetilde{0}\| + \|x-\mathring{x}\| + \|x-\widetilde{x}\| + 2M\right] \\ &\le \rho_1 \left[\|\widetilde{0}\| + \|x-\widetilde{x}\| + 4M\right], \end{aligned} \tag{5}$$

where the last line comes from the upper bounds on $\|\mathring{0}\|, \|x-\mathring{x}\|$ given by A_3.

To bound $\|\widetilde{0}\| + \|x-\widetilde{x}\|$, first note that by the definition of $\widetilde{D}_\lambda$ and since $\mathring{0}$ and $\mathring{x}$ are connected in C_∞,

$$\widetilde{D}_\lambda(0,x) = d_{G_\lambda}(\widetilde{0},\widetilde{x}) + M(\|\widetilde{0}\| + \|x-\widetilde{x}\|) \le d_{G_\lambda}(\mathring{0},\mathring{x}) + M(\|\mathring{0}\| + \|x-\mathring{x}\|).$$

Rearranging then gives

$$\begin{aligned} \|\widetilde{0}\| + \|x-\widetilde{x}\| &\le \frac{1}{M}[d_{G_\lambda}(\mathring{0},\mathring{x}) - d_{G_\lambda}(\widetilde{0},\widetilde{x})] + \|\mathring{0}\| + \|x-\mathring{x}\| \\ &\le \frac{1}{M}[d_{G_\lambda}(\mathring{0},\mathring{x}) - d_{G_\lambda}(\widetilde{0},\widetilde{x})] + 2M. \end{aligned} \tag{6}$$

where the last bound again comes from the event A_3.

Combining (5) with (6) gives

$$\begin{aligned}\|\widetilde{0}\| + \|x - \widetilde{x}\| &\leq \frac{1}{M}\left[d_{G_\lambda}(\mathring{0}, \mathring{x}) - d_{G_\lambda}(\widetilde{0}, \widetilde{x})\right] + 2M \\ &\leq \frac{\rho_1}{M}\left[\|\widetilde{0}\| + \|x - \widetilde{x}\|\right] + 4\rho_1 + 2M,\end{aligned}$$

and rearranging gives

$$\begin{aligned}\|\widetilde{0}\| + \|x - \widetilde{x}\| &\leq \left(1 - \frac{\rho_1}{M}\right)^{-1}(4\rho_1 + 2M) \\ &\leq 8\rho_1 + 4M.\end{aligned}$$

This last inequality holds whenever $\|x\|$ is sufficiently large so that $\rho_1/M < 1/2$. Plugging this last bound into (5) then gives

$$d_{G_\lambda}(\mathring{0}, \mathring{x}) - d_{G_\lambda}(\widetilde{0}, \widetilde{x}) \leq 8\rho_1^2 + 8\rho_1 M.$$

Note also that by (6) we have

$$d_{G_\lambda}(\mathring{0}, \mathring{x}) - d_{G_\lambda}(\widetilde{0}, \widetilde{x}) \geq M(\|\widetilde{0}\| - \|\mathring{0}\| + \|x - \widetilde{x}\| - \|x - \mathring{x}\|) \geq 0,$$

where to get the second inequality we used the definition of $\mathring{0}, \mathring{x}$ together with the fact that (by our connectivity assumption) $\widetilde{0}, \widetilde{x} \in C_\infty$.

Thus, putting everything together, we see that on the event A,

$$\begin{aligned}\left|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)\right| &= \left|d_{G_\lambda}(\mathring{0}, \mathring{x}) - d_{G_\lambda}(\widetilde{0}, \widetilde{x}) - M\left[\|\widetilde{0}\| + \|x - \widetilde{x}\|\right]\right| \\ &\leq d_{G_\lambda}(\mathring{0}, \mathring{x}) - d_{G_\lambda}(\widetilde{0}, \widetilde{x}) + M\left[\|\widetilde{0}\| + \|x - \widetilde{x}\|\right] \\ &\leq \left(8\rho_1^2 + 8\rho_1 M\right) + \left(8\rho_1 M + 4M^2\right) = O\left(M^2\right) = o\left(\|x\|\right),\end{aligned}$$

and so we will have shown (4) once we justify our initial assumption that $\widetilde{0}, \widetilde{x}, \mathring{0}, \mathring{x}$ are all connected.

Claim 10. *On the event A, the points $\widetilde{0}$ and $\widetilde{x}$ lie in C_∞. In particular, $\mathring{0} \xleftrightarrow{G_\lambda} \widetilde{0}$ and $\mathring{x} \xleftrightarrow{G_\lambda} \widetilde{x}$.*

Proof of Theorem 10: On the event A_2, there are no large finite components intersecting G_λ. Thus it suffices to show that $\widetilde{0}$ and $\widetilde{x}$ lie in a component of size at least $\|x\|/2$.

By the triangle inequality,

$$d_{G_\lambda}(\widetilde{0}, \widetilde{x}) \geq \|\widetilde{0} - \widetilde{x}\| \geq \|x\| - [\|\widetilde{0}\| + \|x - \widetilde{x}\|].$$

Using (6)[4] and the trivial bound $d_{G_\lambda}(\widetilde{0}, \widetilde{x}) \geq 0$,

$$\begin{aligned}\|\widetilde{0}\| + \|x - \widetilde{x}\| &\leq \frac{1}{M} d_{G_\lambda}(\mathring{0}, \mathring{x}) + 2M \\ &\leq \frac{\rho_1}{M}\|\mathring{0} - \mathring{x}\| + \rho_1 + 2M \\ &\leq \frac{\rho_1}{M}\left[\|x\| + \|\mathring{0}\| + \|x - \mathring{x}\|\right] + \rho_1 + 2M \\ &\leq \frac{\rho_1}{M}\|x\| + 3\rho_1 + 2M.\end{aligned}$$

The inequality on the second line follows from the chemical distance condition given by A_1, and the fourth line comes from A_3.

Thus, we have

$$\begin{aligned}d_{G_\lambda}(\widetilde{0}, \widetilde{x}) &\geq \|x\| - \left[\frac{\rho_1}{M}\|x\| + 3\rho_1 + 2M\right] \\ &= \left(1 - \frac{\rho_1}{M}\right)\|x\| - 3\rho_1 - 2M\end{aligned}$$

[4]Note that (6) was derived without assuming Theorem 10.

$$\geq \frac{1}{2}\|x\|,$$

where the last line holds as long as $\|x\|$ is sufficiently large. Thus $\widetilde{0}$ and $\widetilde{x}$ lie in a component of size at least $\|x\|/2$; since A_2 holds, they lie in the infinite component and $\widetilde{0}, \mathring{0}, \widetilde{x}$, and $\mathring{x}$ are all connected in G_λ.

■

Finally, we show that A^c can be ignored:

Claim 11.

$$\mathbb{E}\left[\mathbb{1}_{A^c}\left|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)\right|\right] = o(\|x\|).$$

Proof of Theorem 11: By the triangle inequality and Cauchy-Schwarz, we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{A^c}|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)|\right] &\leq \mathbb{E}[\mathbb{1}_{A^c} D_\lambda(0,x)] + \mathbb{E}[\mathbb{1}_{A^c}\widetilde{D}_\lambda(0,x)] \\ &\leq \sqrt{\mathbb{P}\left(A^c\right)}\left(\sqrt{\mathbb{E}[D_\lambda(0,x)^2]} + \sqrt{\mathbb{E}\left[\widetilde{D}_\lambda(0,x)^2\right]}\right).\end{aligned}$$

Since, by Theorem 9, $\sqrt{\mathbb{P}\left(A^c\right)} = o(M^{-1})$, it suffices to show that $\mathbb{E}\left[D_\lambda(0,x)^2\right], \mathbb{E}\left[\widetilde{D}_\lambda(0,x)^2\right] = O(M^2\|x\|^2)$. We have a deterministic bound $\widetilde{D}_\lambda(0,x)^2 = O\left(M^2\|x\|^2\right)$, so it remains to consider $\mathbb{E}D_\lambda(0,x)^2$. Taking C_1, C_2, c_3 as in Theorem 7, we have

$$\begin{aligned}\mathbb{E}[D_\lambda(0,x)^2] &\leq C_1^2\|x\|^2 + \sum_{t=\lfloor C_1\|x\|\rfloor}^{\infty}(t+1)^2\mathbb{P}(D_\lambda(0,x) > t) \\ &\leq C_1^2\|x\|^2 + C_2\sum_{t=\lfloor C_1\|x\|\rfloor}^{\infty}(t+1)^2\exp(-c_3 t^{.99}) = O(\|x\|^2),\end{aligned}$$

as desired.

■

Combining (4) and Theorem 11 then gives

$$\mathbb{E}[|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)|] = \mathbb{E}[\mathbb{1}_A|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)|] + \mathbb{E}[\mathbb{1}_{A^c}|D_\lambda(0,x) - \widetilde{D}_\lambda(0,x)|] = o(\|x\|),$$

which completes our proof of Theorem 8. □

5. Bounding the derivative by a sum along the geodesic

We now use Russo's formula to bound the derivative of $\mathbb{E}\widetilde{D}_\lambda(0,x)$. Define f so that $f(X_\lambda) = \widetilde{D}_\lambda(0,x)$. For $q \in X_\lambda$, define $\Delta_q f(X_\lambda) = f(X_\lambda \setminus \{q\}) - f(X_\lambda)$; note that for all q, $\Delta_q f(X_\lambda) \geq 0$. Since f only depends on X_λ restricted to the finite-volume set $R(\widetilde{D}) := B(0, M\|x\|) \cap B(x, M\|x\|)$, Russo's formula, Theorem 24, tells us that

$$\frac{d}{d\lambda}\mathbb{E}[f(X_\lambda)] = -\frac{1}{\lambda}\mathbb{E}\left[\sum_{q\in X_\lambda \cap R(\widetilde{D})}\Delta_q f(X_\lambda)\right].$$

There exist a few versions of a Russo-type formula for Poisson processes, for example see [20], [21], [28], and [19]; these are related to but not identical to the form we give. We leave the details of our proof in the appendix.

Recall that we define $\widetilde{\pi}$ to be the random open path realizing $\widetilde{D}_\lambda(0,x)$, that is, $\widetilde{\pi}$ is an open path in G_λ between points $\widetilde{0}, \widetilde{x} \in G_\lambda$ with

$$\widetilde{D}_\lambda(0,x) = |\widetilde{\pi}| + M(\|\widetilde{0}\| + \|x - \widetilde{x}\|).$$

The main result of this section is the following:

Proposition 12.

$$\mathbb{E}\left[\sum_{q\in X_\lambda\cap R(\widetilde{D})}\Delta_q f(X_\lambda)\right]\le\mathbb{E}\left[\sum_{q\in\widetilde{\pi}}\mathbb{1}\left\{{G_\lambda\setminus\{q\}\atop p(q)\leftrightarrow s(q)}\right\}d_{G_\lambda\setminus\{q\}}(p(q),s(q))\right]+\lambda^2 o(\|x\|),$$

where here $p(q)$ represents the vertex of $\widetilde{\pi}$ preceding q and $s(q)$ represents the vertex of $\widetilde{\pi}$ following q.

Proof. First, note that if $q\notin\widetilde{\pi}$ then we have $\Delta_q f(X_\lambda)=0$ since in this case removing q cannot increase $\widetilde{D}_\lambda(0,x)$. Thus we have

$$\mathbb{E}\left[\sum_{q\in X_\lambda\cap R(\widetilde{D})}\Delta_q f(X_\lambda)\right]=\mathbb{E}\left[\sum_{q\in\widetilde{\pi}}\Delta_q f(X_\lambda)\right].$$

To bound the terms $\Delta_q f(X_\lambda)$, we first define the following "deletion-tolerant" uniqueness event:

$$A':=\left\{\begin{array}{c}\forall q\in X_\lambda\cap R(\widetilde{D}),(G_\lambda\setminus\{q\})\cap(q+[-\frac{M}{2},\frac{M}{2}]^d)\text{ has at most}\\ \text{one component of metric diameter at least }(M/2)-2\end{array}\right\}.$$

Then we claim

Claim 13.

$$\mathbb{E}\left[\mathbb{1}_{A'}\left(\sum_{q\in\widetilde{\pi}}\Delta_q f(X_\lambda)\right)\right]\le\lambda o(\|x\|)+\mathbb{E}\left[\sum_{\substack{q\in\widetilde{\pi}\\ \setminus B(\widetilde{0},M)\\ \setminus B(\widetilde{x},M)}}\mathbb{1}\left\{{G_\lambda\setminus\{q\}\atop p(q)\leftrightarrow s(q)}\right\}d_{G_\lambda\setminus\{q\}}(p(q),s(q))\right],$$

where here $p(q)$ represents the vertex of $\widetilde{\pi}$ preceding q and $s(q)$ represents the vertex of $\widetilde{\pi}$ following q.

Proof of Theorem 13. We first bound $\Delta_q f(X_\lambda)$ for q near $\widetilde{0}$ and $\widetilde{x}$. We claim that if $q\in B(\widetilde{0},M)\cap\widetilde{\pi}$, then $\Delta_q f(X_\lambda)\le M(M+1)$. To see this, consider the subpath π' of $\widetilde{\pi}$ that starts at $s(q)$ and continues to $\widetilde{x}$. We see that $|\pi'|\le|\widetilde{\pi}|$ and $\|s(q)\|\le\|\widetilde{0}\|+M+1$, so that

$$\begin{aligned}f(X_\lambda\setminus\{q\})&\le M\|s(q)\|+|\pi'|+M\|\widetilde{x}-x\|\\&\le M(M+1+\|\widetilde{0}\|)+|\widetilde{\pi}|+M\|\widetilde{x}-x\|\\&=M(M+1)+f(X_\lambda),\end{aligned}$$

which gives the desired bound on $\Delta_q f(X_\lambda)$. The same argument gives the same bound whenever q lies in $B(\widetilde{x},M)\cap\widetilde{\pi}$. Thus we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{1}_{A'}\left(\sum_{q\in\widetilde{\pi}\cap(B(\widetilde{0},M)\cup B(\widetilde{x},M))}\Delta_q f(X_\lambda)\right)\right]&\le\mathbb{E}[M(M+1)|X_\lambda\cap(B(\widetilde{0},M)\cup B(\widetilde{x},M))|]\\&=\lambda O(M^{d+2})=\lambda o(\|x\|).\end{aligned}$$

Thus, to prove Theorem 13, it only remains to show that for any $q\in\widetilde{\pi}\setminus B(\widetilde{0},M)\setminus B(\widetilde{x},M)$, we have

$$\mathbb{1}_{A'}\Delta_q f(X_\lambda)\le\mathbb{1}\left\{{G_\lambda\setminus\{q\}\atop p(q)\leftrightarrow s(q)}\right\}d_{G_\lambda\setminus\{q\}}(p(q),s(q)).$$

First, we note that if A' holds and $q\in\widetilde{\pi}$ but $\|q-\widetilde{0}\|,\|q-\widetilde{x}\|>M$, then $p(q)$ is connected to $s(q)$ in $G_\lambda\setminus\{q\}$. This is because if they were disconnected, the subpath of $\widetilde{\pi}$ from $\widetilde{0}$ to $p(q)$ and the subpath of $\widetilde{\pi}$ from $s(q)$ to $\widetilde{x}$ would lie in two different connected components of $q+[-\frac{M}{2},\frac{M}{2}]^d$ which both have large diameter, contradicting A'.

Now, assuming $p(q)$ is connected to $s(q)$ in $G_\lambda \setminus \{q\}$, we see that we have a path π' of length at most $|\widetilde{\pi}| + d_{G_\lambda \setminus \{q\}}(p(q), s(q))$ from $\widetilde{0}$ to $\widetilde{x}$ in $G_\lambda \setminus \{q\}$ obtained by composing with a geodesic from $p(q)$ to $s(q)$ in $G_\lambda \setminus \{q\}$. Thus, we have the desired bound.

■

The last step to showing Theorem 12 is to show that A' is very likely, that is:

Claim 14.

$$\mathbb{E}\left[\mathbb{1}_{(A')^c}\left(\sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda)\right)\right] = \lambda^2 o(\|x\|).$$

Proof of Theorem 14. Since we have almost-sure bounds $0 \leq f(X_\lambda) \leq M\|x\|$, we see that

$$\sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda) \leq M\|x\| |X_\lambda \cap R(\widetilde{D})|.$$

$|X_\lambda \cap R(\widetilde{D})|$ is Poisson with parameter $\lambda \mathrm{vol}(R(\widetilde{D})) = O(\lambda M^d \|x\|^d)$, and so by Cauchy-Schwarz we have

$$\mathbb{E}\left[\mathbb{1}_{(A')^c}\left(\sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda)\right)\right] \leq \lambda \sqrt{\mathbb{P}((A')^c)} O(M^{d+1}\|x\|^{d+1}).$$

To bound $\mathbb{P}((A')^c)$, we use Theorem 2 and Theorem 5 to compute

$$\begin{aligned}
\mathbb{P}((A')^c) &\leq \mathbb{E}\left[\sum_{q\in R(\widetilde{D})} \mathbb{1}\left\{\begin{array}{c}(G_\lambda \setminus \{q\}) \cap (q + [-\frac{M}{2}, \frac{M}{2}]^d) \text{ has more than one} \\ \text{component of metric diameter at least } (M/2) - 2\end{array}\right\}\right] \\
&= \lambda \int_{q\in R(\widetilde{D})} \mathbb{P}\left(\begin{array}{c}(G(X_\lambda \cup \{q\}) \setminus \{q\}) \cap (q + [-\frac{M}{2}, \frac{M}{2}]^d) \text{ has more than one} \\ \text{component of metric diameter at least } (M/2) - 2\end{array}\right) dq \\
&= \lambda \int_{q\in R(\widetilde{D})} \mathbb{P}\left(\begin{array}{c}G_\lambda \cap (q + [-\frac{M}{2}, \frac{M}{2}]^d) \text{ has more than one} \\ \text{component of metric diameter at least } (M/2) - 2\end{array}\right) dq \\
&= \lambda O(M^d \|x\|^d) O(\exp(-\frac{c'}{2}M)) \\
&= \lambda o(\|x\|^{-2d-3}).
\end{aligned}$$

■

Combining Theorem 13 and Theorem 14 then gives

$$\begin{aligned}
\mathbb{E}\left[\sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda)\right] &= \mathbb{E}\left[\mathbb{1}_{A'} \sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda)\right] + \mathbb{E}\left[\mathbb{1}_{(A')^c} \sum_{q\in X_\lambda \cap R(\widetilde{D})} \Delta_q f(X_\lambda)\right] \\
&\leq \mathbb{E}\left[\sum_{\substack{q\in \widetilde{\pi} \\ \setminus B(\widetilde{0}, M) \\ \setminus B(\widetilde{x}, M)}} \mathbb{1}\left\{\begin{smallmatrix} G_\lambda \setminus \{q\} \\ p(q) \leftrightarrow s(q)\end{smallmatrix}\right\} d_{G_\lambda \setminus \{q\}}(p(q), s(q))\right] + \lambda^2 o(\|x\|) \\
&\leq \mathbb{E}\left[\sum_{q\in \widetilde{\pi}} \mathbb{1}\left\{\begin{smallmatrix} G_\lambda \setminus \{q\} \\ p(q) \leftrightarrow s(q)\end{smallmatrix}\right\} d_{G_\lambda \setminus \{q\}}(p(q), s(q))\right] + \lambda^2 o(\|x\|),
\end{aligned}$$

as desired. □

6. Lattice animals of dependent weight

We now bound the expression on the right hand side of Theorem 12. To this end, we apply the greedy lattice animal bounds of Can, Nakajima, and Nguyen [3]. Though the original statement of the following lemma is in terms of edges, there is no problem in instead considering random variables indexed by vertices.

Lemma 15 (Lemma 2.7 in [3])**.** *Let $(Y_{\mathbf{z}})_{\mathbf{z}\in\mathbb{Z}^d}$ be a family of $\mathbb{N}$-valued random variables indexed by $\mathbb{Z}^d$. Let $A \geq 1$. Suppose that for each $N \geq 1$ and each AN-separated subset S of $\mathbb{Z}^d$, the family $\{Y_{\mathbf{z}} = N\}_{\mathbf{z}\in S}$ is independent. Define $q_N := \sup_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}(Y_{\mathbf{z}} = N)$. Suppose for some $B < \infty$ we have $\sum_{N=0}^{\infty} N^{d+2} q_N^{1/d} \leq B$. Then there exists C depending only on d, A, and B such that the following holds. For any random path Π in $\mathbb{Z}^d$ starting from 0 such that every vertex of the path has l^∞-distance 1 from the preceding vertex, and any $L \in \mathbb{N}$ we have*

$$\mathbb{E}[\sum_{\mathbf{z}\in\Pi} Y_{\mathbf{z}}] \leq CL + C\sum_{\ell\geq L} \ell\mathbb{P}(|\Pi| = \ell)^{1/2}$$

The proof of Theorem 15 is exactly the same as the proof of Lemma 2.7 in [3].

In order to apply this to our setting, we will have to bound our desired quantity (which is a sum over random points in $\mathbb{R}^d$) by a sum over a random subset of $\mathbb{Z}^d$. To do this, first, for each $\mathbf{z} \in \mathbb{Z}^d$, denote by $Q(\mathbf{z})$ the unit cube $\mathbf{z} + [-\frac{1}{2}, \frac{1}{2}]^d$ centered at $\mathbf{z}$. Let $\widetilde{\Pi} \subset \mathbb{Z}^d$ be the set of $\mathbf{z} \in \mathbb{Z}^d$ such that the cube $Q(\mathbf{z})$ intersects $\widetilde{\pi}$. Recall that for each point q in the geodesic $\widetilde{\pi}$ we denote the point preceding q in $\widetilde{\pi}$ by $p(q)$ and the point succeeding q by $s(q)$. We then can write:

$$\sum_{q\in\widetilde{\pi}} \mathbb{1}\left\{ {G_\lambda\setminus\{q\} \atop p(q)\leftrightarrow s(q)} \right\} d_{G_\lambda\setminus\{q\}}(p(q), s(q)) = \sum_{\mathbf{z}\in\widetilde{\Pi}} \sum_{q\in\widetilde{\pi}\cap Q(\mathbf{z})} \mathbb{1}\left\{ {G_\lambda\setminus\{q\} \atop p(q)\leftrightarrow s(q)} \right\} d_{G_\lambda\setminus\{q\}}(p(q), s(q)).$$

Although this gives us a sum over a random path in $\mathbb{Z}^d$, the summands do not possess the independence property required by Theorem 15; therefore we construct a slightly different variable in order to employ the method of [3]. For this, we have to recall (a version of) the construction used to prove Lemma 3.4 in [27] (and in Antal-Pisztora [1] to prove the main theorem).

For a scale $0 < R < \infty$ and $z \in \mathbb{R}^d$, consider the event

$$U_{z,R} := \left\{ \begin{array}{c} z + [-R, R]^d \text{ is connected to } z + ([-4.9R, 4.9R]^d)^c, \\ z + [-5R, 5R] \text{ contains at most one component of diameter at least } R/5 \end{array} \right\}.$$

Lemma 16. *Let $\lambda_0 > \lambda_c$. Given $\varepsilon > 0$, there exists $R_0 = R_0(\varepsilon, \lambda_0, d)$ such that for all $R \geq R_0$,*

$$\sup_{\lambda\geq\lambda_0} \mathbb{P}(U_{z,R}^c) \leq \varepsilon$$

for every $z \in \mathbb{R}^d$.

Proof. This follows immediately from Theorem 3 and Theorem 5. □

For $\varepsilon > 0$ to be fixed later, take $R(\varepsilon, \lambda_0)$ as above. Also fix an "origin" $\mathbf{z} \in \mathbb{Z}^d$. Then, for $w \in \mathbb{Z}^d$, call w *white* if $U_{\mathbf{z}+Rw,R}$ holds, and call w *black* otherwise. Say that $w, w' \in \mathbb{Z}^d$ are $*$-adjacent if they have l^∞-distance one, i.e. for each $i = 1, ..., d$, $|w_i - w_i'| \leq 1$.

Let S be the set of $a \in \mathbb{Z}^d$ $*$-adjacent to 0. For $a \in \mathbb{Z}^d$, we denote by $\mathfrak{C}_a^*$ the $*$-connected component of black sites containing a, where recall that w is black if $U_{\mathbf{z}+Rw}^c$ holds. Let $\mathcal{C}(\mathbf{z}) := \{0\} \cup \bigcup_{a\in S} \mathfrak{C}_a^*$, let $\overline{\mathcal{C}}(\mathbf{z})$ be the set of sites either contained in $\mathcal{C}(\mathbf{z})$ or $*$-adjacent to an element of $\mathcal{C}(\mathbf{z})$, and then take $W(\mathbf{z}) = \bigcup_{w\in\overline{\mathcal{C}}(\mathbf{z})} (\mathbf{z} + Rw + [-5R, 5R]^d)$. The key property of this construction is the following:

Lemma 17. *If $p, q \in \mathbf{z} + [-R/2, R/2]^d$ are connected in a graph G which agrees with G_λ outside of $\mathbf{z} + [-R, R]^d$, then they are connected by a path in $G \cap W(\mathbf{z})$.*

Proof. The argument is the same as that from [1] (see also [14] for an argument using homology). □

We then use the following observation, which roughly states that edge-geodesics lying inside a set have length at most a constant times the volume of the set.

Proposition 18. *Given $d \geq 2$, there exists $C_d < \infty$ such that the following holds. If G is an induced subgraph of G_λ, for any $W \subset \mathbb{R}^d$ and any path γ in G, there exists a self-avoiding path γ' in G with the same endpoints as γ such that all vertices of γ' lie in γ, and*

$$|\gamma' \cap W| \leq C_d \mathrm{vol}(W + B(1/2)).$$

Moreover, we can either take $\gamma = \gamma'$ or $|\gamma'| < |\gamma|$.

Proof. First note that we can construct γ' with the same endpoints as γ with the property that any pair of vertices in $\gamma' \cap W$ which are not adjacent in γ' have distance larger than 1; we do so as follows. If γ already has this property, take $\gamma' = \gamma$. Otherwise, take γ, choose a pair of non-adjacent vertices $x, y \in \gamma$ with $\|x - y\| \leq 1$, and replace the subpath of γ from the first instance of x to the last instance of y with the edge $e = \{x, y\}$ (which is in G). This modification strictly decreases the length of γ, and it decreases the number of pairs of non-adjacent vertices with distance at most one by at least one; it also does not change the endpoints of γ. After inductively applying this procedure finitely many times, we will have γ' with the desired property.

Next, there exists a 1-separated vertex subset $S \subset W \cap \gamma'$ such that $|S| \geq \frac{1}{2}|W \cap \gamma'|$; simply take S to be a subset of maximal size among subsets which contain no pair of vertices adjacent in γ'.

Then we have the inclusion of disjoint balls

$$\bigsqcup_{s \in S} B\left(s, \frac{1}{2}\right) \subseteq W + B\left(0, \frac{1}{2}\right)$$

which implies the volume bound

$$|S| \mathrm{vol}\left(B_d\left(1/2\right)\right) \leq \mathrm{vol}(W + B(1/2)).$$

So we have

$$|\gamma' \cap W| \leq 2|S| \leq \frac{2}{\mathrm{vol}(B_d(1/2))} \mathrm{vol}(W + B(1/2)),$$

and then we can take $C_d := 2/\mathrm{vol}(B_d(1/2))$, which depends only on d, and we are done. □

Now we can use all of the above to construct a variable $Y_{\mathbf{z}}$ which satisfies the hypotheses of Theorem 15 and which bounds our desired quantity up to a constant factor.

Proposition 19. *Let R be fixed. For each $\mathbf{z} \in \mathbb{Z}^d$, define the random variable*

$$Y_{\mathbf{z}} := |\overline{\mathcal{C}}(\mathbf{z})|$$

where $\overline{\mathcal{C}}(\mathbf{z})$ is as defined above. Then there exists some constant $C'_{d,R} < \infty$ depending only on d and R such that

$$\sum_{q \in \tilde{\pi}} \mathbb{1}\left\{ {G_\lambda \setminus \{q\} \atop p(q) \leftrightarrow s(q)} \right\} d_{G_\lambda \setminus \{q\}}(p(q), s(q)) \leq C'_{d,R} \sum_{\mathbf{z} \in \tilde{\Pi}} Y_{\mathbf{z}}.$$

Proof. To avoid clutter, for three points $p, q, s \in G_\lambda$ let us define

$$Z_{p,q,s} := \mathbb{1}\left\{ {G_\lambda \setminus \{q\} \atop p \leftrightarrow s} \right\} d_{G_\lambda \setminus \{q\}}(p, s).$$

Then the quantity we want to bound is

$$\sum_{q \in \tilde{\pi}} Z_{p(q),q,s(q)} = \sum_{\mathbf{z} \in \tilde{\Pi}} \sum_{q \in \tilde{\pi} \cap Q(\mathbf{z})} Z_{p(q),q,s(q)}$$

By Theorem 17, if p, s adjacent to $q \in Q(\mathbf{z})$ are connected in $G_\lambda \setminus \{q\}$, then they are connected by a path γ in $W(\mathbf{z})$. By Theorem 18, we then have a path γ' from p to s in $G_\lambda \setminus \{q\} \cap W$ with

$$|\gamma'| \leq C_d \mathrm{vol}(W + B_d(1/2))$$

$$\leq C_d|\overline{\mathcal{C}}(\mathbf{z})|(10R+1)^d.$$

This gives a bound for each $Z_{p,q,s}$, and therefore we have

$$\begin{aligned}\sum_{q\in\widetilde{\pi}} Z_{p(q),q,s(q)} &\leq \sum_{\mathbf{z}\in\widetilde{\Pi}} \sum_{q\in\widetilde{\pi}\cap Q(\mathbf{z})} C_d(10R+1)^d|Y_{\mathbf{z}}| \\ &= C_d(10R+1)^d \sum_{\mathbf{z}\in\widetilde{\Pi}} |\widetilde{\pi}\cap Q(\mathbf{z})||Y_{\mathbf{z}}|.\end{aligned}$$

But since $\widetilde{\pi}$ is a minimal edge path from $\widetilde{0}$ to $\widetilde{x}$, Theorem 18 applied to $\widetilde{\pi}$ shows that

$$|\widetilde{\pi}\cap Q(\mathbf{z})| \leq C_d \mathrm{vol}(Q(\mathbf{z}) + B_d(1/2)) \leq 2^d C_d.$$

Combining all of the above then gives

$$\sum_{q\in\widetilde{\pi}} Z_{p(q),q,s(q)} \leq 2^d C_d^2 (10R+1)^d \sum_{\mathbf{z}\in\widetilde{\Pi}} |Y_{\mathbf{z}}|,$$

and we are done. □

Now we establish that the family $(Y_{\mathbf{z}})_{\mathbf{z}\in\mathbb{Z}^d}$ satisfies the hypotheses of Theorem 15.

Lemma 20. *For each $R < \infty, d \geq 2$ there exists $A < \infty$ depending only on d and R such that the following holds. For any $N \geq 1$ and any AN-separated subset $S \subset \mathbb{Z}^d$, the family of events $\{Y_{\mathbf{z}} = N\}_{\mathbf{z}\in S}$ is independent.*

Proof. For a fixed R, the event $\{Y_{\mathbf{z}} = |\overline{\mathcal{C}}(\mathbf{z})| = N\}$ is witnessed by a connected set in $\mathbb{Z}^d$ containing 0 with size at most N, and the state of each site $w \in \mathbb{Z}^d$ is determined by the restriction of X_λ to $\mathbf{z} + Rw + [-5R, 5R]^d$. Therefore this event depends only on X_λ restricted to $B(\mathbf{z}, 10dRN)$.

Therefore, if $S \subset \mathbb{Z}^d$ is a $(30dRN)$-separated subset, the events $\{Y_{\mathbf{z}} = N\}_{\mathbf{z}\in S}$ all depend on X_λ restricted to pairwise disjoint regions, and hence are independent. Taking $A = 30dR$, we are done. □

Lemma 21. *Let $\lambda_0 > \lambda_c$. Then there exist $R < \infty$ and $B < \infty$ depending only on d and λ_0 such that the following holds. If we define $Y_{\mathbf{z}}$ as in Theorem 19 with parameter R, then for all $\lambda \geq \lambda_0$ we have*

$$\sum_{N=0}^{\infty} N^{d+2} q_N^{1/d} \leq B,$$

where

$$q_N := \sup_{\mathbf{z}\in\mathbb{Z}^d} \mathbb{P}^\lambda(Y_{\mathbf{z}} = N).$$

Proof. First let us establish a bound on the tail of $Y_{\mathbf{z}}$ for a sufficiently large choice of R.

First, set $\delta = 1/(4\cdot 3^d)$, so in particular $0 < \delta < 1/(2\cdot 3^d)$. Then, using domination by product measures ([23], Theorem 0.0.(ii)), choose $\varepsilon > 0$ sufficiently small that any $11d$-dependent field on $\mathbb{Z}^d$ with $\mathbb{P}(w \text{ is white}) \geq 1-\varepsilon$ dominates an independent site percolation on $\mathbb{Z}^d$ with $\mathbb{P}(w \text{ is white}) = 1-\delta$. We then choose $R(\varepsilon, \lambda_0, d)$ as in Theorem 16 and perform the construction of $\overline{\mathcal{C}}(\mathbf{z})$ with parameter R. In particular, we note that by our choice of R, for any $a \in \mathbb{Z}^d$, $|\mathfrak{C}_a^*|$ is stochastically dominated by the law of the size of the $*$-connected cluster of black sites at the origin of a Bernoulli site percolation with $\mathbb{P}(0 \text{ is black}) = \delta$.

Now, to bound the tail of $Y_{\mathbf{z}}$, note that

$$Y_{\mathbf{z}} = |\overline{\mathcal{C}}(\mathbf{z})| \leq 3^d|\mathcal{C}(\mathbf{z})| \leq 3^d(1 + \sum_{a\in S} |\mathfrak{C}_a^*|),$$

so a union bound gives

$$\begin{aligned}\mathbb{P}(Y_{\mathbf{z}} \geq N) &\leq \mathbb{P}(\sum_{a\in S} |\mathfrak{C}_a^*| \geq (N/3^d) - 1) \\ &\leq \sum_{a\in S} \mathbb{P}(|\mathfrak{C}_a^*| \geq (N/3^{2d}) - 3^{-d})\end{aligned}$$

$$\leq 3^d \sum_{k=\lceil (N/3^{2d})-1 \rceil}^{\infty} [(2\cdot 3^d)\delta]^k,$$

where the last bound estimates the tail of the size of an independent Bernoulli site percolation cluster using the fact that the number of $*$-connected subsets of $\mathbb{Z}^d$ of size k containing the origin is at most $(2\cdot 3^d)^k$. Now, by our choice of δ, this is decaying exponentially in N; let us say

$$\mathbb{P}(Y_{\mathbf{z}} \geq N) \leq Ce^{-cN},$$

where C, c now only depend on d.

Hence we compute

$$\sum_{N=0}^{\infty} N^{d+2} q_N^{1/d} \leq C \sum_{N=0}^{\infty} N^{d+2} e^{-(c/d)N} \leq B,$$

where B is a function of C, c, and d, and hence actually only depends on d. □

Finally, to make use of Theorem 15 effectively we must bound the tails of $|\widetilde{\Pi}|$. As a technical point, in order to apply Theorem 15, we need a random path starting from 0, so we define a new random path $\widehat{\Pi}$ by $\gamma \cup \widetilde{\Pi}$, where γ is a shortest path in $\mathbb{Z}^d$ from 0 to the first point of $\widetilde{\Pi}$.

Lemma 22. *There exist constants C_1, C_2, c_3 depending only on λ_0 and d such that for all $t \geq C_1\|x\|$ we have*

$$\mathbb{P}(\widehat{\Pi} \geq t) \leq C_2 \exp(-c_3 t^{.99}).$$

Proof. First, we can use the crude bound

$$|\widetilde{\Pi}| \leq |\widetilde{\pi}|$$

and

$$|\gamma| \leq \sqrt{d}(\|\widetilde{0}\| + 1)$$

to get (assuming that $\|x\|$ and hence M is sufficiently large)

$$|\widehat{\Pi}| \leq \sqrt{d}(\|\widetilde{0}\| + 1) + |\widetilde{\pi}| \leq M\|\widetilde{0}\| + |\widetilde{\pi}| + M\|x - \widetilde{x}\| + \sqrt{d} = \widetilde{D}_\lambda(0, x) + \sqrt{d}.$$

Thus, it suffices to bound the tails of $\widetilde{D}_\lambda(0, x)$. But since we also have the bound $\widetilde{D}_\lambda(0, x) \leq D_\lambda(0, x) + M(\|\mathring{0}\| + \|x - \mathring{x}\|)$, our desired statement now follows directly from Theorem 7. □

Finally we can get the desired linear bound on our random sum:

Lemma 23. *Let $\lambda_0 > \lambda_c$. Choose R as in Theorem 21 and define $Y_{\mathbf{z}}$ as in Theorem 19 with parameter R. Then there exists $B' < \infty$ depending only on λ_0 and d such that for every $\lambda \geq \lambda_0$ we have*

$$\mathbb{E}^\lambda \left[\sum_{\mathbf{z} \in \widetilde{\Pi}} Y_{\mathbf{z}} \right] \leq B' \|x\|$$

whenever $\|x\| \geq 1$.

Proof. Since the summands are all nonnegative and by construction $\widetilde{\Pi} \subset \widehat{\Pi}$, it suffices to show the same bound with $\widetilde{\Pi}$ replaced by $\widehat{\Pi}$.

Choosing R and B based on λ_0 and d as in Theorem 21, Theorem 20 provides us with a parameter A so that the family $\{Y_{\mathbf{z}}\}_{\mathbf{z} \in \mathbb{Z}^d}$ satisfies the hypotheses of Theorem 15. Taking C_1 as in Theorem 22 and setting $L = C_1\|x\|$, applying Theorem 15 then gives

$$\mathbb{E}^\lambda \left[\sum_{\mathbf{z} \in \widetilde{\Pi}} Y_{\mathbf{z}} \right] \leq C \left(C_1\|x\| + C_2 \sum_{t=\lfloor C_1\|x\| \rfloor}^{\infty} \exp(-c_3 t^{.99}) \right),$$

for some C only depending on λ_0 and d (through A and B). Then we can choose B' depending only on C, C_1, C_2, c_3 (hence only on λ_0 and d) such that the above is bounded by $B'\|x\|$ whenever $\|x\| \geq 1$. □

7. Lipschitz continuity of the time constant

Finally, we can prove Theorem 1.

Proof of Theorem 1. First let us prove a Lipschitz bound for $\widetilde{D}_\lambda$. Recall that, since $\widetilde{D}_\lambda(0,x)$ is bounded and only depends on X_λ restricted to the finite volume region $R(\widetilde{D}) \subset \mathbb{R}^d$, its expectation is differentiable by Theorem 24. Fixing $\lambda_0 > \lambda_c$, for any $\lambda_0 \leq \lambda < \lambda' < \infty$, we then have

$$\begin{aligned}
0 &\leq \mathbb{E}\widetilde{D}_\lambda(0,x) - \mathbb{E}\widetilde{D}_{\lambda'}(0,x) \\
&= \int_\lambda^{\lambda'} -\frac{d}{d\ell}\mathbb{E}\widetilde{D}_\ell(0,x)d\ell \\
&= \int_\lambda^{\lambda'} \frac{1}{\ell}\mathbb{E}\left[\sum_{q\in X_\ell \cap R(\widetilde{D})} \Delta_q f(X_\ell)\right] d\ell \\
&\leq \int_\lambda^{\lambda'} \frac{1}{\ell}\mathbb{E}\left[\sum_{q\in\widetilde{\pi}} \mathbb{1}\left\{\substack{G_\ell\setminus\{q\} \\ p(q)\leftrightarrow s(q)}\right\} d_{G_\ell\setminus\{q\}}(p(q),s(q))\right] d\ell + \left(\int_\lambda^{\lambda'} \ell d\ell\right) o(\|x\|) \\
&\leq \int_\lambda^{\lambda'} \frac{1}{\ell} C'_{d,R}\mathbb{E}^\ell\left[\sum_{\mathbf{z}\in\widetilde{\Pi}} Y_{\mathbf{z}}\right] d\ell + (\lambda')^2 o(\|x\|) \\
&\leq (C'_{d,R}/\lambda_0)B'\|x\||\lambda' - \lambda| + (\lambda')^2 o(\|x\|).
\end{aligned}$$

The equality in the third line is from Theorem 24 (recalling the definitions of f and Δ_q from the beginning of Section 5); the fourth line is from Theorem 12. In the fifth line, we have chosen R based on λ_0 and d as specified by Theorem 21, and defined $Y_{\mathbf{z}}$ with parameter R as in Theorem 19; the fifth line then follows from Theorem 19, and the final bound from Theorem 23.

Therefore we have

$$\begin{aligned}
0 \leq \mu_\lambda - \mu_{\lambda'} &= \lim_{x\to\infty} \frac{\mathbb{E}D_\lambda(0,x) - \mathbb{E}D_{\lambda'}(0,x)}{\|x\|} \\
&= \lim_{x\to\infty} \frac{\mathbb{E}\widetilde{D}_\lambda(0,x) - \mathbb{E}\widetilde{D}_{\lambda'}(0,x)}{\|x\|} \\
&\leq \limsup_{x\to\infty} \frac{(C'_{d,R}/\lambda_0)B'|\lambda' - \lambda|\|x\| + (\lambda')^2 o(\|x\|)}{\|x\|} \\
&= (C'_{d,R}/\lambda_0)B'|\lambda' - \lambda|.
\end{aligned}$$

The second line is from Theorem 8, and the third line is the last inequality we derived. Thus, the function $\lambda \mapsto \mu_\lambda$ is Lipschitz on $[\lambda_0, \infty)$ with constant $(C'_{d,R}/\lambda_0)B'$.

Note also that, if instead of using the bound $(1/\ell) \leq (1/\lambda_0)$ above, we compute the integral, we obtain a bound of $C'_{d,R}B'\log(\lambda'/\lambda)$. Hence μ_λ is Lipschitz as a function of *the logarithm of* λ, that is, $t \mapsto \mu_{e^t}$ is Lipschitz in t on $[t_0, \infty)$ for $t_0 > \log\lambda_c$. □

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Appendix A. A Russo-type formula

Throughout this section, denote by X_λ a Poisson point process of intensity λ on K, where K is a finite volume Borel subset of $\mathbb{R}^d$, and let $f : \{F \subset K : F \text{ finite }\} \to \mathbb{R}$ be a bounded measurable function.

Proposition 24 (A Russo-type formula)**.** *For every $\lambda > 0$, $\frac{d}{d\lambda}\mathbb{E}f(X_\lambda)$ exists and we have*

$$\frac{d}{d\lambda}\mathbb{E}f(X_\lambda) = \frac{1}{\lambda}\mathbb{E}\left[\sum_{x\in X_\lambda}[f(X_\lambda) - f(X_\lambda \setminus \{x\})]\right].$$

Proof. First, we show that $\mathbb{E}f(X_\lambda)$ is differentiable in λ. To see this, first note that, since K has finite volume, X_λ can be sampled by first sampling $N \sim \text{Poi}(\lambda|K|)$ and then letting $X_\lambda = U_{[N]}$, where $U_1, U_2, ...$ is a sequence of independent random variables sampled according to the uniform measure on K and $U_{[k]} := \bigcup_{i=1}^k \{U_i\}$. Therefore we can write

$$\begin{aligned}\mathbb{E}f(X_\lambda) &= \sum_{k=0}^\infty \mathbb{E}[f(X_\lambda)|N = k]\mathbb{P}(N = k)\\ &= \sum_{k=0}^\infty \mathbb{E}[f(U_{[k]})]e^{-\lambda|K|}\frac{(\lambda|K|)^k}{k!}.\end{aligned}$$

Note that $\mathbb{E}[f(U_{[k]})]$ is independent of λ, so each term of this series is differentiable in λ. Moreover, (since $\mathbb{E}f(U_{[k]}) \le \|f\|_\infty$) it is straightforward to check that the sum of the termwise derivatives converges uniformly on compact subsets of $[0,\infty)$; therefore $\mathbb{E}f(X_\lambda)$ is differentiable on $(0,\infty)$ as desired.

Now recall the following coupling for X_λ and $X_{\lambda-\varepsilon}$: Given $\eta \in (0,1)$, let $X_{\lambda,\eta}$ be the random subset of X_λ where each element of X_λ is independently retained with probability η and deleted with probability $1-\eta$. Then $X_{\lambda,\eta}$ is a Poisson point process on K with intensity $\eta\lambda$ and $X_\lambda \setminus X_{\lambda,\eta}$ is a Poisson point process on K with intensity $(1-\eta)\lambda$. In particular, if we take $\eta = \frac{\varepsilon}{\lambda}$, we see that $X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}}$ is equal in distribution $X_{\lambda-\varepsilon}$. Therefore we have

$$\begin{aligned}\frac{d}{d\lambda}\mathbb{E}f(X_\lambda) &= \lim_{\varepsilon\to 0^+}\frac{\mathbb{E}f(X_\lambda) - \mathbb{E}f(X_{\lambda-\varepsilon})}{\varepsilon}\\ &= \lim_{\varepsilon\to 0^+}\frac{\mathbb{E}[f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})]}{\varepsilon}.\end{aligned}$$

Next, note that

$$\begin{aligned}\mathbb{E}\left[|f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})|\mathbb{1}_{\{|X_{\lambda,\frac{\varepsilon}{\lambda}}|\ge 2\}}\right] &\le 2\|f\|_\infty\mathbb{P}(|X_{\lambda,\frac{\varepsilon}{\lambda}}| \ge 2)\\ &= 2\|f\|_\infty\mathbb{P}(|X_\varepsilon| \ge 2)\\ &= O(\varepsilon^2).\end{aligned}$$

Therefore, in order to compute the desired limit, it suffices to consider

$$\mathbb{E}\left[[f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})]\mathbb{1}_{\{|X_{\lambda,\frac{\varepsilon}{\lambda}}|\le 1\}}\right] = \mathbb{E}\left[[f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})]\mathbb{1}_{\{|X_{\lambda,\frac{\varepsilon}{\lambda}}|=1\}}\right],$$

where for the above equality we used the fact that $f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}}) = 0$ on the event $\left\{|X_{\lambda,\frac{\varepsilon}{\lambda}}| = 0\right\}$.

Now, conditioning on X_λ we can compute

$$\begin{aligned}\mathbb{E}\left[[f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})]\mathbb{1}_{\{|X_{\lambda,\frac{\varepsilon}{\lambda}}|=1\}}\middle| X_\lambda\right] &= \sum_{S\subset X_\lambda}(f(X_\lambda) - f(X_\lambda \setminus S))\mathbb{1}_{\{|S|=1\}}\mathbb{P}(X_{\lambda,\frac{\varepsilon}{\lambda}} = S|X_\lambda)\\ &= \left(\frac{\varepsilon}{\lambda}\right)\left(1-\frac{\varepsilon}{\lambda}\right)^{|X_\lambda|-1}\sum_{x\in X_\lambda}[f(X_\lambda) - f(X_\lambda \setminus \{x\})].\end{aligned}$$

Thus, we have

$$\frac{d}{d\lambda}\mathbb{E}f(X_\lambda) = \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\mathbb{E}\left[[f(X_\lambda) - f(X_\lambda \setminus X_{\lambda,\frac{\varepsilon}{\lambda}})]\mathbb{1}_{\{|X_{\lambda,\frac{\varepsilon}{\lambda}}|=1\}}\right]$$

$$= \lim_{\varepsilon \to 0^+} \frac{1}{\lambda} \mathbb{E}\left[\left(1 - \frac{\varepsilon}{\lambda}\right)^{|X_\lambda|-1} \sum_{x \in X_\lambda} [f(X_\lambda) - f(X_\lambda \setminus \{x\})] \right]$$

$$= \frac{1}{\lambda} \mathbb{E}\left[\sum_{x \in X_\lambda} [f(X_\lambda) - f(X_\lambda \setminus \{x\})] \right],$$

where the last equality follows from the Dominated Convergence Theorem, since the random variables being integrated are dominated by the integrable random variable $4\|f\|_\infty |X_\lambda|$ for $0 < \varepsilon < \lambda/2$. □

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